l_p-UNITARY MATRICES

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Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Given $p \in \mathbb{R} \cup \{\infty\}$, $p \geqslant 1$, consider Hölder's norms l_p distinct from the Euclidean norm, i.e., suppose $p \neq 2$. We will say that an $n \times n$ matrix U with elements belonging to $\mathbb{K}$ is l_p-unitary if U is an isometry for this norm:

$$\|Ux\|_p = \|x\|_p \qquad \text{for all} \quad x \in \mathbb{K}^n.$$

Let us denote by $\alpha_k(U)$ the singular values of matrix U with respect to norm l_p in the sense of J. F. Maitre and N. H. Vinh; briefly, the l_p, l_p-singular values of U are

$$\alpha_k(U) = \min_{G_{n-k+1}} \max_{\substack{x \in G_{n-k+1} \\ x \neq 0}} \frac{\|Ux\|_p}{\|x\|_p}, \qquad k = 1,2,\dots,n, \tag{1}$$

where G_{n-k+1} ranges over the set of subspaces of $\mathbb{K}^n$ with dimension $n-k+1$ [6]. The number of nonnull l_p, l_p-singular values of a matrix coincides with its rank. It is obvious that the $n \times n$ l_p-unitary matrices have n l_p, l_p-singular values equal to unity, and moreover, they are the unique square matrices of order n with those l_p, l_p-singular values. Therefore, the set $\mathcal{U}_p$ formed by the $n \times n$ l_p-unitary matrices constitutes a subgroup of the linear group GL$(n, \mathbb{K})$.

The objective of this note is to give a complete description of group $\mathcal{U}_p$, for the real and complex cases, simultaneously.

THEOREM 1. *The multiplicative group $\mathcal{U}_p$ of the $n \times n$ l_p-unitary matrices with elements in $\mathbb{K}$ does not depend on p, $p \neq 2$, and it is constituted by the matrices of the form*

$$U = (u_{kj}) = (\varepsilon_k \delta_{k\sigma(j)})$$

with $\varepsilon_k \in \mathbb{K}$ of modulus equal to one, σ a permutation of $\{1,2,\dots,n\}$, and

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δ_{kh} *Kronecker's delta. In other terms,*

$$\mathcal{U}_p = \Big\{ DP \in \mathbb{K}^{n\times n} \Big| D = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n),\ \varepsilon_k \in \mathbb{K}, |\varepsilon_k| = 1,\ k = 1,2,\dots,n,$$

$$P \textit{ is an } n \times n \textit{ permutation matrix } \Big\}.$$

In the proof of this result, which has been exposed in [8], we use properties of the singular values defined in (1). For the maximum norm, l_∞, and its dual norm, l_1, we obtain a second proof in a straightforward manner, first for 2×2 matrices and then for the $n \times n$ general case.

Theorem 1 has a long history, and it has been proved in a number of manners. Thus, in the classical reference [1], by S. Banach, the isometries of spaces $l^{(p)}$ are determined. The author considers the infinite-dimensional case, but the proofs are valid for finite dimensions. Later, in [7, Theorem 7.7 and Example 1 of Section 7.9], by H. Schneider and R. Turner, Theorem 1 is found as a particular case of a theorem on decomposition of isometries for absolute norms in $\mathbb{C}^n$. This decomposition theorem is found again in [9, Theorem 10], by I. Vidav, in a more general context about isometries of finite-dimensional complex Banach spaces.

The problem of searching a complete system of invariants, and canonical forms, for the similarity relation of matrices restricted to a subgroup of $\mathrm{GL}(n,\mathbb{K})$, is still open for the subgroup $\mathcal{U}_p$. (Problems of this class were proposed by C. R. Johnson in 1982 [3].) It is easy to prove that the l_p, l_p-singular values (1) are a (noncomplete) system of invariants of this relation. Now, as we have just seen, the group $\mathcal{U}_p$ contains the group of permutation matrices and, at the same time, is contained in the group of unitary matrices. Canonical forms for the similarity with respect to these two extreme groups are known [2, 4, 5]. Perhaps it is possible to draw indications to the solution of this problem from that fact. In the case of real matrices, the group $\mathcal{U}_p$ is finite ($|\mathcal{U}_p| = 2^n n!$), and it is equal to the group of permutation matrices, except for the sign; so in this case the above-stated problem is essentially combinatorial in nature.

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AN APPLICATION OF ZONAL POLYNOMIALS TO THE GENERATION OF PROBABILITY DISTRIBUTIONS

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1. *Introduction*

The group representation theory of $\mathrm{Gl}(m, R)$, the general linear group, can be used to define the zonal polynomials. From a technical point of view it is quite difficult to define the zonal polynomials. The group-theoretic construction of zonal polynomials can be seen in Farrell (1976) and James (1961, 1964, 1968, 1973, 1976).

The approach to zonal polynomials through group representation theory is as follows. Let V_k be the vector space of homogeneous polynomials $\Phi(X)$ of degree k in the $n = m(m+1)/2$ different elements of the $m \times m$ symmetric matrix X. Corresponding to any congruence transformation $X \to LXL'$, $L \in \mathrm{Gl}(m, R)$, a linear transformation of the space V_k can be defined by $\Phi \to T(L)\Phi : (T(L)\Phi)(X) = \Phi(L^{-1}XL^{-1\prime})$. A representation of $\mathrm{Gl}(m, R)$ in the vector space V_k is defined by this transformation, that is, the mapping $L \to T(L)$ is a homomorphism from $\mathrm{Gl}(m, R)$ to the group of linear transformations of V_k, $T(L_1L_2) = T(L_1)T(L_2)$.

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